Interval Fuzzy Model Identification Using l_∞-Norm

Igor Škrjanc, Sašo Blažič, and Osvaldo Agamennoni

***Abstract*—In this paper, we present a new method of interval fuzzy model identification. The method combines a fuzzy identification methodology with some ideas from linear programming theory. We consider a finite set of measured data, and we use an optimality criterion that minimizes the maximum estimation error between the data and the proposed fuzzy model output. The idea is then extended to modeling the optimal lower and upper bound functions that define the band which contains all the measurement values. This results in lower and upper fuzzy models or a fuzzy model with a set of lower and upper parameters. The model is called the interval fuzzy model (INFUMO). We also showed that the proposed structure uniformly approximates the band of any nonlinear function. The interval fuzzy model identification is a methodology to approximate functions by taking into account a finite set of input and output measurements. This approach can also be used to compress information in the case of large amount of data and in the case of robust system identification. The method can be efficiently used in the case of the approximation of the nonlinear functions family. If the family is defined by a band containing the whole measurement set, the interval of parameters is obtained as the result. This is of great importance in the case of nonlinear circuits' modeling, especially when the parameters of the circuits vary within certain tolerance bands.**

***Index Terms*—Fuzzy model, interval fuzzy model (INFUMO), linear programming, min–max optimization.**

I. Introduction

The problem of a function approximation from a finite set of measured data using an optimality criterion that minimizes the estimation error has received a great deal of attention in the scientific community, especially with the advent of neural network techniques. Continuous piecewise linear (PWL) functions have also been used for a function approximation, particularly since the introduction of the canonical expression [1] and [2]. Since then a high-level canonical piecewise linear (HLCPWL) representation of all the continuous PWL mappings defined over a simplicial partition of a domain in an n-dimensional space has been introduced in [3] and [4]. This representation is able to uniformly approximate any Lipschitz continuous function defined on a compact domain. Moreover, in contrast to neural networks, if the Lipschitz constant of the nonlinear function is known, it is possible to calculate the number of terms required to obtain a given error. An upper and lower PWL function can be evaluated to optimally describe the interval of all the possible values of the uncertain function. A salient feature of this methodology is that the approximation problem is reduced to a linear programming (LP) problem, for which efficient solution algorithms exist in [5].

The fuzzy model, in Takagi–Sugeno (TS) form, approximates the nonlinear system by smoothly interpolating affine local models [6] and [7]. Each local model contributes to the global model in a fuzzy subset of the space characterized by a membership function. This present paper is inspired by the idea that by using a proper evaluation of triangular shape membership functions it is possible to emulate the simplicial HLCPWL approximation technique [4] by following a fuzzy logic approach. The paper focuses on the development of an interval l_∞-norm function approximation methodology problem using the LP technique and the TS fuzzy logic approach. This results in lower and upper fuzzy models or a fuzzy model with lower and upper parameters. This model is called the interval fuzzy model (INFUMO) and it has been shown that the proposed structure uniformly approximates the band of any nonlinear function. It is well known that the structure and shape of if-part fuzzy sets have a significant effect on the fuzzy model approximation of continuous functions [7], [8]. In this case, the proposed approach will exhibit an extra degree of flexibility in the domain partition as well as in the use of different membership functions compared with the HLCPWL technique.

The interval fuzzy model identification is a methodology to approximate functions of a finite set of input and output measurements that can also be used to compress information in the case of the nonlinear function family approximation [9] to obtain the interval of parameters that results in a band containing the whole measurement set. This is of great importance in many technological areas, e.g., nonlinear circuits modeling, especially when the parameters of the circuit vary within a certain tolerance band. Some authors [10]–[12] have proposed models where the expert knowledge is approximated by upper and lower possibility distributions. In the present paper, the bounds are modeled independently of each other.

The paper is organized as follows. In Section II, the background to fuzzy modeling is given. In Section III, the idea of fuzzy model identification using l_∞ norm is described, in Section IV the interval fuzzy model identification is introduced and in Section V an application to the approximation of continuous functions is given.

II. A Nonlinear Model Described in Fuzzy Form

A typical fuzzy model [6] is given in the form of rules

$$\begin{aligned}\mathbf{R}_j:\ &\text{if } x_{p1} \text{ is } \mathbf{A}_{1,k_1} \text{ and } x_{p2} \text{ is } \mathbf{A}_{2,k_2}\\ &\text{and} \ldots \text{and } x_{pq} \text{ is } \mathbf{A}_{q,k_q} \text{ then } y=\phi_j(\mathbf{x})\\ &\qquad j=1,\ldots,m \quad k_1=1,\ldots,f_1\\ &k_2=1,\ldots,f_2 \quad \ldots \quad k_q=1,\ldots,f_q. \quad (1)\end{aligned}$$

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I. Škrjanc and S. Blažič are with the Faculty of Electrical Engineering, University of Ljubljana, 1000 Ljubljana, Slovenia (e-mail: igor.skrjanc@fe.uni-lj.si).

O. Agamennoni is with the Universidad Nacional del Sur, Bahia Blanca 8000, Argentina.

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The q-element vector $\mathbf{x}_p^T = [x_{p1}, \dots, x_{pq}]$ denotes the input or variables in premise, and the variable y is the output of the model. With each variable in premise $x_{pi} (i = 1, \dots, q)$, f_i fuzzy sets $(\mathbf{A}_{i,1}, \dots, \mathbf{A}_{i,f_i})$ are connected, and each fuzzy set $\mathbf{A}_{i,k_i} (k_i = 1, \dots, f_i)$ is associated with a real-valued function $\mu_{A_{i,k_i}}(x_{pi}) : \mathbb{R} \to [0,1]$ that produces the membership grade of the variable x_{pi} with respect to the fuzzy set $\mathbf{A}_{i,k_i}$. To make the list of fuzzy rules complete, all possible variations of the fuzzy sets are given in (1), yielding the number of fuzzy rules $m = f_1 \times f_2 \times \cdots \times f_q$. The variables x_{pi} are not the only inputs of the fuzzy system. Implicitly, the n-element vector $\mathbf{x}^T = [x_1, \dots, x_n]$ also represents an input to the system. This vector is usually referred to as the consequence vector. The functions $\phi_j(\cdot)$ can in general, be arbitrary smooth functions, although linear or affine functions are usually used.

The system in (1) can be described in closed form if the intersection of the fuzzy sets is previously defined. The generalized form of the intersection is the so-called *triangular norm* (T-norm). In our case, the latter was chosen as an algebraic product yielding the output of the fuzzy system shown in (2) at the bottom of the page. It should be noted that there is a slight abuse of notation in (2), since j is not explicitly defined as a running index. From (1), it is evident that each j corresponds to the specific variation of the indexes $k_i, i = 1, \dots, q$.

To simplify (2), a partition of unity is considered where the functions $\beta_j(\mathbf{x}_p)$ defined by (3) at the bottom of the page give information about the fulfilment of the respective fuzzy rule in the normalized form. It is obvious that $\sum_{j=1}^{m} \beta_j(\mathbf{x}_p) = 1$ irrespective of $\mathbf{x}_p$ as long as the denominator of $\beta_j(\mathbf{x}_p)$ is not equal to zero (that can be easily prevented by stretching the membership functions over the whole potential area of $\mathbf{x}_p$). Combining (2) and (3) and changing the summation over k_i to a summation over j we arrive at the following equation:

$$\hat{y} = \sum_{j=1}^{m} \beta_j(\mathbf{x}_p) \phi_j(\mathbf{x}). \tag{4}$$

From (4), it is evident that the output of a fuzzy system is a function of the premise vector $\mathbf{x}_p$ (q-dimensional) and the consequence vector $\mathbf{x}$ (n-dimensional). The dimension of the input space may be lower than $(q + n)$, since it is very usual to have the same variables present in vectors $\mathbf{x}_p$ and $\mathbf{x}$. If a vector $\mathbf{z}$ is constructed in the following manner:

$$\mathbf{z} = \begin{bmatrix} \mathbf{x}_p^T & \mathbf{x}'^T \end{bmatrix}^T = [z_1, z_2, \dots, z_d]^T \tag{5}$$

where $\mathbf{x}'$ consists of those elements of $\mathbf{x}$ that are not present in $\mathbf{x}_p$, the following statement can be made about the dimensionality d of the input space:

$$\max(q, n) \le d \le q + n. \tag{6}$$

The fuzzy system described by (1) or (4) can be seen as a mapping from $\mathbb{R}^d$ to $\mathbb{R}$.

The class of fuzzy models have the form of linear models, this refers to $\{\beta^j\}$ as a set of basis functions. The use of membership functions in the input space with overlapping receptive fields provides interpolation and extrapolation.

Very often, the output value is defined as a linear combination of consequence states

$$\phi_j(\mathbf{x}) = \boldsymbol{\theta}_j^T \mathbf{x}, \qquad j = 1, \dots, m \quad \boldsymbol{\theta}_j^T = [\theta_{j1}, \dots, \theta_{jn}]. \tag{7}$$

If the TS model of the 0th order is chosen, $\phi_j(\mathbf{x}) = \theta_{j0}$, and in the case of the first-order model, the consequent is $\phi_j(\mathbf{x}) = \theta_{j0} + \boldsymbol{\theta}_j^T \mathbf{x}$. Both cases can be treated by the model (7) by adding 1 to the vector $\mathbf{x}$ and augmenting vector $\boldsymbol{\theta}$ with θ_{j0}. To simplify the notation, only the model in (7) will be treated in the rest of this paper. If the matrix of the coefficients for the whole set of rules is written as $\boldsymbol{\Theta}^T = [\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_m]$ and the vector of membership values as $\boldsymbol{\beta}^T(\mathbf{x}_p) = [\beta^1(\mathbf{x}_p), \dots, \beta^m(\mathbf{x}_p)]$, then (4) can be rewritten in the matrix form

$$\hat{y} = \boldsymbol{\beta}^T(\mathbf{x}_p) \boldsymbol{\Theta} \mathbf{x}. \tag{8}$$

The fuzzy model in the form given in (8) is referred to as the affine TS model and can be used to approximate any arbitrary function that maps the compact set $\mathbf{C} \subset \mathbb{R}^d$ to $\mathbb{R}$ with any desired degree of accuracy [7], [13], [14]. The generality can be proved by the Stone–Weierstrass [15] theorem, which indicates that any continuous function can be approximated by a fuzzy basis function expansion [16].

III. Fuzzy Model Identification Using l_∞ Norm

In this section, we discuss an approach to the model parameter estimation where the l_∞ norm is used as the criterion for the measure of the modeling error. We assume a set of premise vectors $\mathbf{X_p} = \{\mathbf{x}_{p1}, \mathbf{x}_{p2}, \dots, \mathbf{x}_{pN}\}$ and a set of antecedent (or consequence) vectors $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$. Assuming (5), a set $\mathbf{Z} = \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N\}$ can be constructed that represents the input measurement data, collected from the compact

$$\hat{y} = \frac{\sum_{k_1=1}^{f_1} \sum_{k_2=1}^{f_2} \cdots \sum_{k_q=1}^{f_q} \mu_{A_{1,k_1}}(x_{p1}) \mu_{A_{2,k_2}}(x_{p2}) \cdots \mu_{A_{q,k_q}}(x_{pq}) \phi_j(\mathbf{x})}{\sum_{k_1=1}^{f_1} \sum_{k_2=1}^{f_2} \cdots \sum_{k_q=1}^{f_q} \mu_{A_{1,k_1}}(x_{p1}) \mu_{A_{2,k_2}}(x_{p2}) \cdots \mu_{A_{q,k_q}}(x_{pq})} \tag{2}$$

$$\beta_j(\mathbf{x}_p) = \frac{\mu_{A_{1,k_1}}(x_{p1}) \mu_{A_{2,k_2}}(x_{p2}) \cdots \mu_{A_{q,k_q}}(x_{pq})}{\sum_{k_1=1}^{f_1} \sum_{k_2=1}^{f_2} \cdots \sum_{k_q=1}^{f_q} \mu_{A_{1,k_1}}(x_{p1}) \mu_{A_{2,k_2}}(x_{p2}) \cdots \mu_{A_{q,k_q}}(x_{pq})}, j = 1, \dots, m \tag{3}$$

set $\mathbf{S} \subset \mathbb{R}^d$. A set of corresponding outputs is also defined as $\mathbf{Y} = \{y_1, y_2, \ldots, y_N\}$. The measurements satisfy the nonlinear equation of the system

$$y_i = g(\mathbf{z}_i), \qquad i = 1, \ldots, N. \tag{9}$$

According to the Stone–Weierstrass theorem, for any given real continuous function g on a compact set $\mathbf{U} \subset \mathbf{R}^d$ and arbitrary $\epsilon > 0$, there exists a fuzzy system f such that

$$\sup_{\mathbf{z}_i \in \mathbf{Z}} |f(\mathbf{z}_i) - g(\mathbf{z}_i)| < \epsilon \qquad \forall i. \tag{10}$$

This implies the approximation of any given real continuous function with the fuzzy function from class $\mathcal{F}^d$ defined in (8). However, it should be pointed out that low values of ϵ imply higher values of m that satisfy (10). In the case of an approximation, the error between the measured values and the fuzzy function outputs can be defined as

$$e_i = y_i - f(\mathbf{x}_i) \qquad \forall i. \tag{11}$$

To estimate the optimal parameters of the proposed fuzzy function the minimization of the maximum modeling error

$$\max_{z_i \in \mathbf{Z}} |y_i - f(\mathbf{z}_i)| \tag{12}$$

over the whole input set $\mathbf{Z}$ is performed. This implies the *min-max* optimization method. In the case of the TS model in (8), the minimization of the expression in (12) can be performed in two steps. The first problem is how to minimize the error with respect to $\mathbf{x}_p$. The answer lies in the proper arrangement of the membership functions. This is a well-known problem in fuzzy systems, and it can be solved with a cluster analysis [17]–[19] or with other approaches. The details will not be discussed in this paper. By having the membership functions defined, the structure of the model is known and only the parameters $\boldsymbol{\Theta}$ are to be defined by the *min–max* optimization

$$\boldsymbol{\Theta} = \arg\min_{\boldsymbol{\Theta}} \max_{\mathbf{z}_i \in \mathbf{Z}} |y_i - \boldsymbol{\beta}^T(\mathbf{x}_{pi})\boldsymbol{\Theta}\mathbf{x}_i| \tag{13}$$

or in the equivalent form

$$\boldsymbol{\Theta} = \arg\min_{\boldsymbol{\Theta}} \max_{\mathbf{z}_i \in \mathbf{Z}} \left| y_i - \sum_{j=1}^{m} \beta_j(\mathbf{x}_{pi})\boldsymbol{\theta}_j^T \mathbf{x}_i \right|. \tag{14}$$

Lemma 1: The min–max optimization problem can be solved as the linear programming problem of minimizing λ subject to the inequalities

$$\begin{aligned} y_i - \sum_{j=1}^{m} \beta_j(\mathbf{x}_{pi})\boldsymbol{\theta}_j^T \mathbf{x}_i \le \lambda, \qquad & i = 1, 2, \ldots, N \\ -y_i + \sum_{j=1}^{m} \beta_j(\mathbf{x}_{pi})\boldsymbol{\theta}_j^T \mathbf{x}_i \le \lambda, \qquad & i = 1, 2, \ldots, N \\ & \lambda \ge 0 \end{aligned} \tag{15}$$

on the parameter $\boldsymbol{\theta}_j (j = 1, \ldots, m)$. The resulting λ stands for the maximum approximation error.

Proof: If we define

$$\lambda = \max_{\mathbf{z}_i \in \mathbf{Z}} \left| y_i - \sum_{j=1}^{m} \beta_j(\mathbf{x}_{pi})\boldsymbol{\theta}_j^T \mathbf{x}_i \right| \tag{16}$$

and take into account that $\mathbf{z}_i$ encapsulates the information in vectors $\mathbf{x}_{pi}$ and $\mathbf{x}_i$ [see (5)], this directly implies the following system of inequalities:

$$\left| y_i - \sum_{j=1}^{m} \beta^j \boldsymbol{\theta}_j^T \mathbf{x}_i \right| \le \lambda, \qquad i = 1, 2, \ldots, N \tag{17}$$

which can then be written in the form (15). This concludes the Proof of Lemma 1, and the optimization problem from (14) can be stated as the minimization of λ subject to (15). □

The idea of an approximation can be interpreted as the most representative fuzzy function to describe the domain of outputs $\mathbf{Y}$ as a function of inputs $\mathbf{Z}$. This problem can also be viewed as a problem of data reduction, which often appears in identification problems with large data sets.

IV. Interval Fuzzy Model Identification

In the case of an uncertain nonlinear function that can be defined as a member of the family of functions

$$\mathcal{G} = \{g : \mathbf{S} \to \mathbb{R}^1 \,|\, g(\mathbf{z}) = g_{\text{nom}}(\mathbf{z}) + \Delta g(\mathbf{z})\} \tag{18}$$

where g_{nom} stands for the nominal function and the uncertainty Δg satisfies $\sup_{\mathbf{z} \in \mathbf{S}} |\Delta g(\mathbf{z})| \le c, c \in \mathbb{R}$.

Let us consider a function g that is a member of the class $g \in \mathcal{G}$ and the corresponding set of measured output values $\mathbf{Y} = \{y_1, \ldots, y_N\}$ over the set of inputs $\mathbf{Z}$, i.e., $y_i = g(\mathbf{z}_i), g \in \mathcal{G}, \mathbf{z}_i \in \mathbf{S}, i = 1, \ldots, N$.

The idea of robust interval fuzzy modeling is to find a lower fuzzy function $\underline{f}$ and an upper fuzzy function $\bar{f}$ satisfying

$$\underline{f}(\mathbf{z}_i) \le g(\mathbf{z}_i) \le \bar{f}(\mathbf{z}_i) \qquad \forall \mathbf{z}_i \in \mathbf{S}. \tag{19}$$

In this sense, a function from the class $\mathcal{G}$ can always be found in the band defined by the upper and lower fuzzy functions. The main aim when defining the band is that it is as narrow as possible in accordance with the proposed constraints. The problem has been treated in the literature using the piecewise linear function approximation [3]. Our approach, using the fuzzy function approximation, can be viewed as a generalization of the piecewise linear approach, and it gives a better approximation, or at least a much narrower approximation band.

The upper and lower fuzzy functions, respectively, can be found by solving the following optimization problems:

$$\min_{\underline{f}} \max_{z_i \in \mathbf{Z}} |y_i - \underline{f}(\mathbf{z}_i)| \text{ subject to } y_i - \underline{f}(\mathbf{z}_i) \ge 0 \tag{20}$$

$$\min_{\bar{f}} \max_{z_i \in \mathbf{Z}} |y_i - \bar{f}(\mathbf{z}_i)| \text{ subject to } y_i - \bar{f}(\mathbf{z}_i) \le 0. \tag{21}$$

The solutions to both problems can be found by linear programming, because both problems can be viewed as linear programming problems, as is stated in the following lemma. First of all, we have to define a lower and an upper fuzzy function as $\underline{f}(\mathbf{z}) = \boldsymbol{\beta}^T(\mathbf{x}_p)\underline{\boldsymbol{\Theta}}\mathbf{x}$ and $\bar{f}(\mathbf{z}) = \boldsymbol{\beta}^T(\mathbf{x}_p)\bar{\boldsymbol{\Theta}}\mathbf{x}$.

Lemma 2: The min–max optimization problems in (20) and (21) can be solved as the linear programming problems of minimizing λ_1 and λ_2 subject to the inequalities

$$
\begin{aligned}
&y_i - \sum_{j=1}^{m} \beta_j(\mathbf{x}_{pi})\underline{\boldsymbol{\theta}}_j^T \mathbf{x}_i \le \lambda_1, \qquad i = 1,\ldots,N \\
&y_i - \sum_{j=1}^{m} \beta_j(\mathbf{x}_{pi})\underline{\boldsymbol{\theta}}_j^T \mathbf{x}_i \ge 0, \qquad i = 1,\ldots,N \\
&\lambda_1 \ge 0
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
&-y_i + \sum_{j=1}^{m} \beta_j(\mathbf{x}_{pi})\bar{\boldsymbol{\theta}}_j^T \mathbf{x}_i \le \lambda_2, \qquad i = 1,\ldots,N \\
&y_i - \sum_{j=1}^{m} \beta_j(\mathbf{x}_{pi})\underline{\boldsymbol{\theta}}_j^T \mathbf{x}_i \le 0, \qquad i = 1,\ldots,N \\
&\lambda_2 \ge 0
\end{aligned} \tag{23}
$$

on the parameters $\underline{\boldsymbol{\theta}}_j, \bar{\boldsymbol{\theta}}_j, j = 1,\ldots,m$, and λ_1 and λ_2 that stand for the maximum approximation errors of both approximation functions.

Proof: The proof can be directly inferred from Lemma 1.

Remark 1: Note that $\underline{f}(\mathbf{z}) \le \bar{f}(\mathbf{z})$ does not necessarily hold for the arbitrary $\mathbf{z}$. This can happen in the part of space where no identification data were present. The method does not explicitly check for this, but by choosing a reasonable data set this phenomenon can be avoided implicitly. This weakness quality of the model in the part of space where there is not enough excitation is not something particular to the proposed approach. Rather, it is a well-known property of any identification procedure.

Example 1: Let us define a class $\mathcal{G}$ with $g_{\mathrm{nom}}(z) = \cos(z)\sin(z)$ and the uncertainty $\Delta g(z) = \gamma\cos(8z), 0 \le \gamma \le 0.2$. The functions from the class are defined in the domain $\mathbf{S} = \{z \mid -1 \le z \le 1\}$ and the set of "measurements" is $\mathbf{Z} = \{z_i \mid z_i = 0.021k, k = -47, -46, \ldots, 47\} \subset \mathbf{S}$. In this case, the dimensionality of the input space is $d = 1$, and therefore the premise and the consequent variables are the same as the measurements, i.e., $x_{pi} = x_i = z_i, i = 1,\ldots,N$. For convenience, the independent variable will be denoted by x (or x_i) in the following. The family of functions $\mathcal{G}$ for a few values of γ is presented in Fig. 1.

Two approximation models were used to solve the problem—the first-order fuzzy model and the singleton fuzzy model. They differ according to the size of vector $\mathbf{x}$ in the fuzzy rule consequence and the size of the parameter vector $\boldsymbol{\theta}_j$. In the first approach, the vector has the size of two ($\mathbf{x}^T = [x, 1], \boldsymbol{\theta}_j^T = [\theta_{j1}, \theta_{j0}]$), while in the second approach the vector $\mathbf{x}$ consists only of a constant with the value 1 and $\boldsymbol{\theta}_j = \theta_{j0}$. Our task is to find the upper and lower bounding functions by solving the linear programming problems defined in (22) and (23). In both cases (first-order and singleton model) eight triangular and equidistant membership functions will be used ($m = 8$). They are presented in Fig. 2.

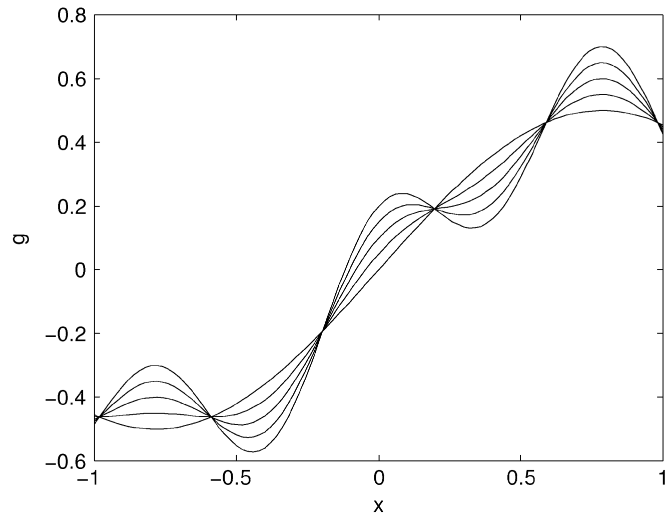

Fig. 1. Family of functions $\mathcal{G}$.

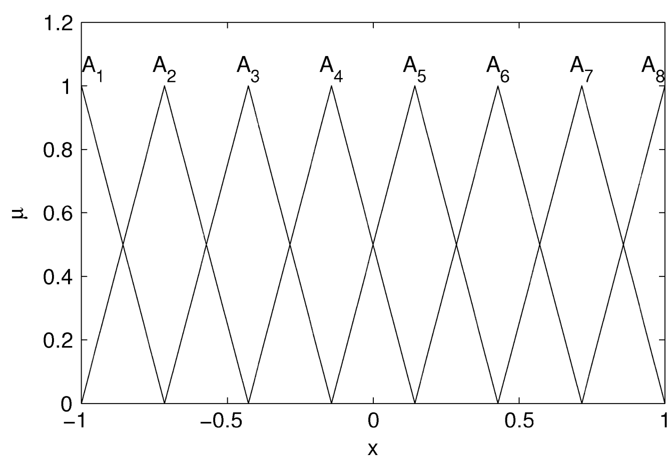

Fig. 2. Membership function set.

In the case of the first-order fuzzy model, the lower and upper fuzzy models take the following form:

$$
\underline{\mathbf{R}}_j : \text{if } x \text{ is } \mathbf{A}_j \text{ then } y = \underline{\theta}_{j1}x + \underline{\theta}_{j0}, \qquad j = 1,\ldots,m \tag{24}
$$

$$
\bar{\mathbf{R}}_j : \text{if } x \text{ is } \mathbf{A}_j \text{ then } y = \bar{\theta}_{j1}x + \bar{\theta}_{j0}, \qquad j = 1,\ldots,m. \tag{25}
$$

The results are shown in Fig. 3, where we have the dotted set of values that belong to the functions set $\mathbf{Y}$ and the solid lines for the lower approximation function $\underline{f}(x) = \boldsymbol{\beta}^T(x)\underline{\boldsymbol{\Theta}}\mathbf{x}$ and the upper approximation function $\bar{f}(x) = \boldsymbol{\beta}^T(x)\bar{\boldsymbol{\Theta}}\mathbf{x}$. In Fig. 4, the approximation errors $\sup_{g\in\mathcal{G}}(\underline{f}(x) - g(x))$ and $\inf_{g\in\mathcal{G}}(\bar{f}(x) - g(x))$ are presented. They show how conservative is the obtained approximation. In other words, they show the difference between the fuzzy approximation band and the tight envelope around the family of functions. Naturally, our wish is to obtain the smallest possible error (ideally, both errors presented in Fig. 4 would be 0).

In the second approach, the consequent vector $\mathbf{x}$ is a constant of dimension 1; without any loss of generality it can be made equal to $1(\mathbf{x} = 1)$. The fuzzy model in this case is called

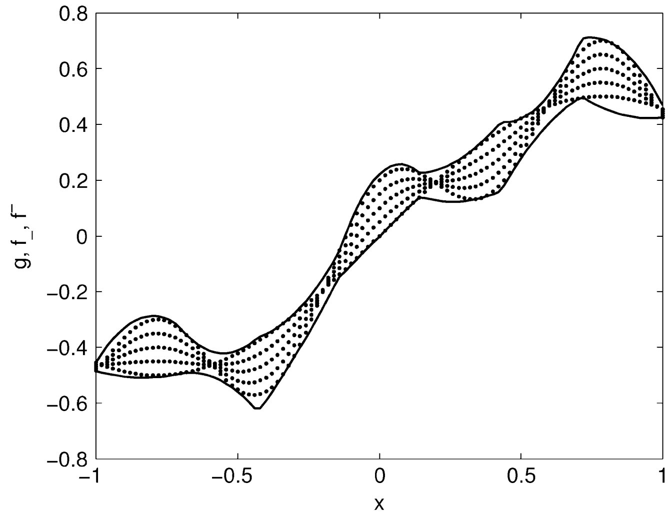

Fig. 3. Data, lower, and upper bound.

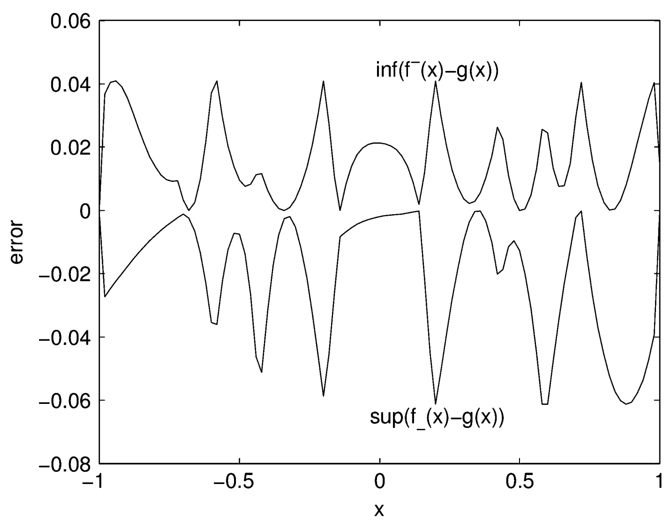

Fig. 4. Difference between the approximation functions and the actual envelope of the family of functions.

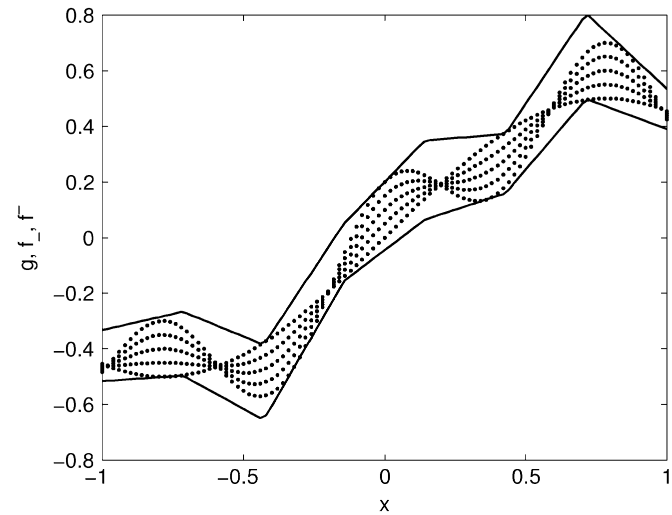

Fig. 5. Data, lower, and upper bound for the singleton model.

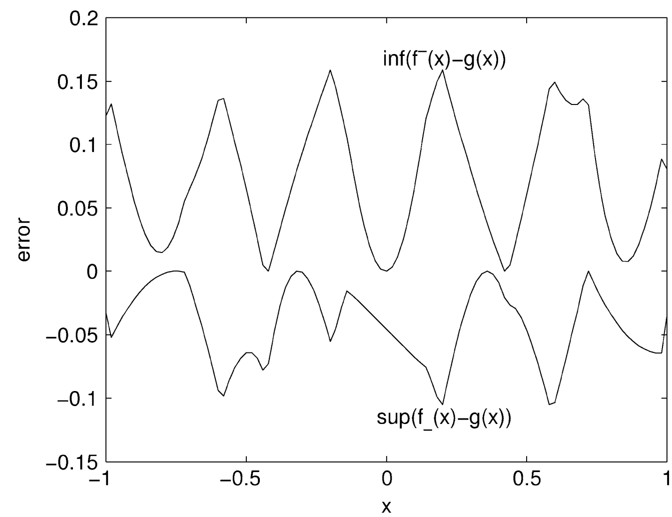

Fig. 6. Difference between the approximation functions and the actual envelope of the family of functions for the singleton model.

the singleton fuzzy model. The lower and upper singleton fuzzy models can be described by the following fuzzy rules:

$$\underline{\mathbf{R}}_j : \text{if } x \text{ is } \mathbf{A}_j \text{ then } y = \underline{\theta}_{j0}, \qquad j = 1, \ldots, m \tag{26}$$

$$\bar{\mathbf{R}}_j : \text{if } x \text{ is } \mathbf{A}_j \text{ then } y = \bar{\theta}_{j0}, \qquad j = 1, \ldots, m. \tag{27}$$

The resulting approximation functions reduce to $\underline{f}(x) = \boldsymbol{\beta}^T(x)\underline{\boldsymbol{\Theta}}$ and $\bar{f}(x) = \boldsymbol{\beta}^T(x)\bar{\boldsymbol{\Theta}}$, and can be obtained by solving the linear programming problem. The results of the treated example are shown in Fig. 5, where the dots represent the set of values $\{\mathbf{X}, \mathbf{Y}\}$, while the solid lines show the lower and upper approximation functions $\underline{f}$ and $\bar{f}$, respectively. In Fig. 6 the approximation errors $\sup_{g\in\mathcal{G}}(\underline{f}(x) - g(x))$ and $\inf_{g\in\mathcal{G}}(\bar{f}(x) - g(x))$ are depicted.

It is obvious that a better approximation is obtained if the first-order fuzzy model is used. Nevertheless, the singleton model can also result in a better approximation if the membership functions are arranged in the "optimal" way. In our example, the focus has been to show the modeling of the uncertainty band for the family of functions. We have assumed a uniform partitioning of the input data interval into eight fuzzy membership sets. A better approximation or a narrower band could be obtained if we determined the membership functions' partitioning using a cluster analysis of the data [17]–[19]. The approximation by means of the c-means clustering algorithm that is used to define the optimal fuzzy partitioning with eleven membership functions is shown in Fig. 7. In Fig. 8, the approximation error is shown. The resulting fuzzy partitioning obtained with the c-means fuzzy clustering algorithm is shown in Fig. 9.

V. Application for the Approximation of Continuous Functions

In this section, an application for the approximation of continuous functions is presented. In the case of higher dimensionality the problem of numerous subspaces arises. The identification of the interval fuzzy model is, in this case, divided into two parts. In the first part, we calculate a classical fuzzy model using the l_2-norm optimization to obtain the parameters of the

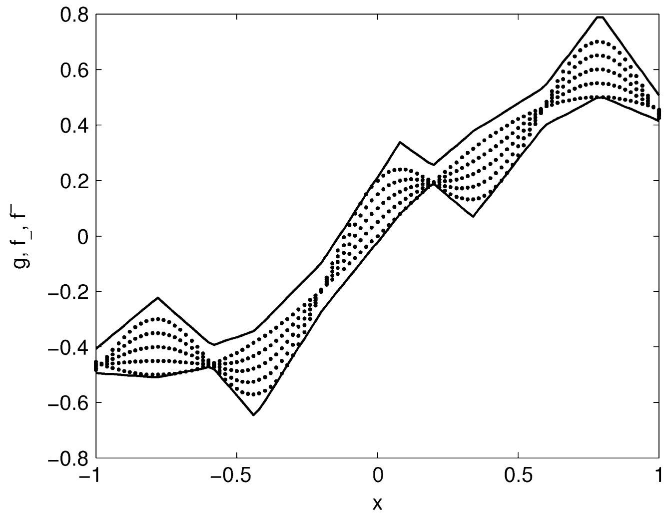

Fig. 7. Data, lower, and upper bound for the second singleton model.

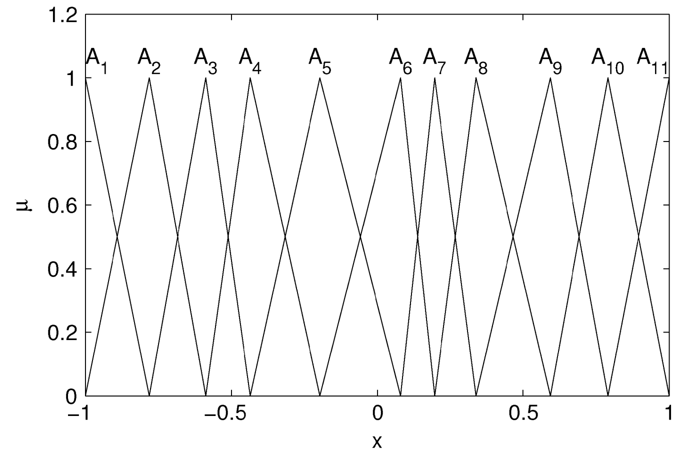

Fig. 9. Membership function set for the second singleton model.

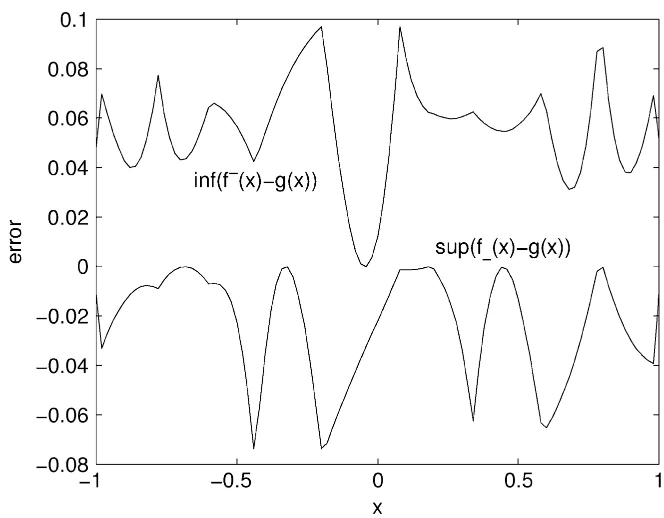

Fig. 8. Difference between the approximation functions and the actual envelope of the family of functions for the second singleton model.

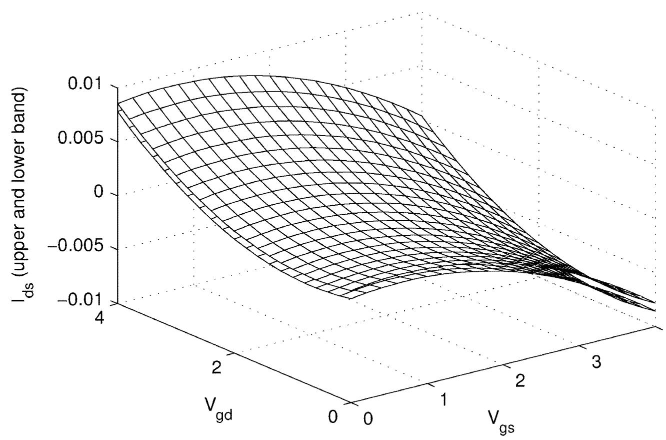

Fig. 10. Upper and lower surfaces of the interval fuzzy model for the Mosfet characteristic.

fuzzy model, and in the second part we use the l_∞-norm optimization to find the optimal lower and upper bounds of the data. This is especially interesting in the case when the data represent families of functions. In our case, the dc-current characteristic $I_{\rm ds}(V_{\rm gd}, V_{\rm gs})$ of a Glasmost n-channel Mosfet transistor is modeled [9]. Equation (28), as shown at the bottom of the page, represents the static behavior of the device, where $\mu = 0.0675\ \mathrm{m^2V^{-1}s^{-1}}, C_{\rm ox} = 1.38 \cdot 10^{-3}\ \mathrm{Fm^{-2}}, V_0 = 1$ V, $W = 20\ \mu$m, $L = 2\ \mu$m, and $V_{\rm th} = 30$ mV. Because the parameters are in tolerance bands and vary, the model results in the family of surfaces. The domain of the function $I_{\rm ds}$ is given by $\mathbf{S} = \{V_{\rm gs}, V_{\rm gd} : 0 \le V_{\rm gs} \le 4, 0 \le V_{\rm gd} \le 4\}$. The whole domain has been divided into four overlapping subspaces and both variables have been divided into two equal subspaces. The rule that is used to model the nonlinear statical characteristic is the following:

$$\mathbf{R}_j : \textbf{if } V_{\rm gs} \textbf{ is } \mathbf{A}_{1,k_1} \textbf{ and } V_{\rm gd} \textbf{ is } \mathbf{A}_{2,k_2} \textbf{ then}$$

$$I_{\rm ds} = a_j V_{\rm gs} + b_j V_{\rm gd} + r_j \tag{29}$$

$$k_1 = 1, 2 \quad k_2 = 1, 2 \quad j = 2k_1 + k_2 - 2. \tag{30}$$

In Fig. 10, the upper and lower bounds of the Mosfet characteristic are presented as an INFUMO model of the following form:

$$\bar{\mathbf{R}}_j : \text{if } V_{\rm gs} \text{ is } \mathbf{A}_{1,k_1} \text{ and } V_{\rm gd} \text{ is } \mathbf{A}_{2,k_2} \text{ then}$$

$$\bar{I}_{\rm ds} = \bar{a}_j V_{\rm gs} + \bar{b}_j V_{\rm gd} + \bar{r}_j \tag{31}$$

$$k_1 = 1, 2 \quad k_2 = 1, 2 \quad j = 2k_1 + k_2 - 2 \tag{32}$$

$$I_{\rm ds} = \begin{cases} \frac{W\mu C_{\rm ox}}{L}(V_{\rm gs} - V_{\rm gd})(V_{\rm gs} + V_{\rm gd} + 2V_{\rm th} - 2V_0), & V_{\rm gs} \ge V_0, V_{\rm gd} \ge V_0 \\ \frac{W\mu C_{\rm ox}}{L}\left((V_{\rm gs} - V_0)(V_{\rm gs} - V_0 + 2V_{\rm th}) - V_{\rm th}^2\left(e^{(V_{\rm gd} - V_0/V_{\rm th})} - 1\right)\right), & V_{\rm gs} \ge V_0, V_{\rm gd} < V_0 \\ \frac{W\mu C_{\rm ox}}{L}\left(V_{\rm th}^2\left(e^{(V_{\rm gd} - V_0/V_{\rm th})} - 1\right) - (V_{\rm gd} - V_0 + 2V_{\rm th})\right), & V_{\rm gs} < V_0, V_{\rm gd} \ge V_0 \\ \frac{W\mu C_{\rm ox}}{L}V_{\rm th}^2\left(e^{(V_{\rm gs} - V_0/V_{\rm th})} - e^{(V_{\rm gd} - V_0/V_{\rm th})}\right), & V_{\rm gs} < V_0, V_{\rm gd} < V_0 \end{cases} \tag{28}$$

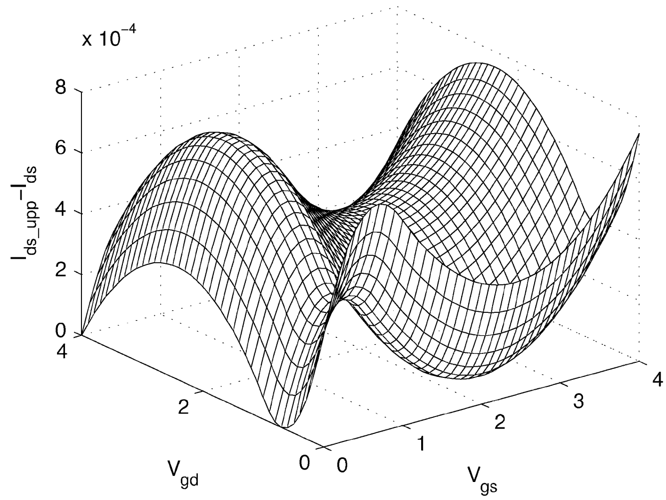

Fig. 11. Difference between the upper surface of the INFUMO model and the nonlinear Mosfet characteristic.

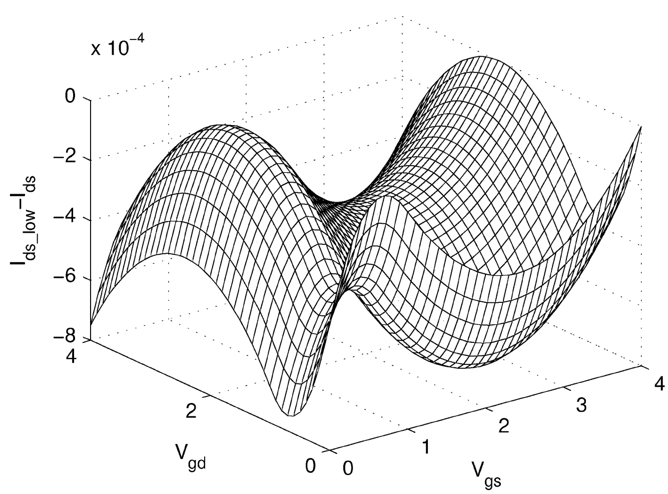

Fig. 12. Difference between the lower surface of the INFUMO model and the nonlinear Mosfet characteristic.

$$\underline{\mathbf{R}}_j : \text{if } V_{\text{gs}} \text{ is } \mathbf{A}_{1,k_1} \text{ and } V_{\text{gd}} \text{ is } \mathbf{A}_{2,k_2} \text{ then}$$
$$\underline{I}_{\text{ds}} = \underline{a}_j V_{\text{gs}} + \underline{b}_j V_{\text{gd}} + \underline{r}_j \tag{33}$$
$$k_1 = 1,2 \quad k_2 = 1,2 \quad j = 2k_1 + k_2 - 2. \tag{34}$$

In Figs. 11 and 12, the errors between the INFUMO approximation and the real nonlinear characteristic are presented.

VI. Conclusion

A new method of interval fuzzy model identification is proposed that is applicable when a finite set of measurement data is available. The method combines a fuzzy identification methodology with some ideas from linear programming theory. The idea is then extended to the modeling of optimal lower and upper bound functions that define the band that contains all the measurement values. This results in lower and upper fuzzy models or the INFUMO. The INFUMO model is of great importance in the case of families of functions where the parameters of the observed system vary in certain intervals due to the tolerance of individual elements. It has been shown that the proposed structure uniformly approximates the band of any nonlinear family of functions. Some potential areas where the INFUMO model could be used are fault detection, robust system identification, robust control, and compressing the information in the case of large amounts of data.

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Igor Škrjanc received the B.Sc., M.Sc., and Ph.D. degrees, all in electrical engineering, from the Faculty of Electrical and Computer Engineering, University of Ljubljana, Slovenia, in 1988, 1991, and 1996, respectively.

He is currently an Associate Professor with the same faculty. His main research interests are in adaptive, predictive, fuzzy, and fuzzy adaptive control systems.

Sašo Blažič received the B.Sc., M.Sc., and Ph.D. degrees from the Faculty of Electrical Engineering, University of Ljubljana, Slovenia, in 1996, 1999, and 2002, respectively.

Currently, he is an Assistant Professor with the same faculty. His main interests include adaptive, predictive, and fuzzy control with a stress on the robustness issues of adaptive control.

Osvaldo Enrique Agamennoni was born in Bahía Blanca, Argentina, on December 21, 1953. He received the Electrical Engineering degree and the Ph.D. degree in Control System, both from the Universidad Nacional del Sur (UNS), Bahía Blanca, Argentina, in 1979 and 1991, respectively.

From 1992 to 1994, he was a Postdoctoral Fellow in the Chemical Engineering Department, the University of Sydney, Australia. Since 1995, he has been an Associate Professor in the Departamento de Ingeniería Eléctrica y de Computadoras de la UNS. He is the Head of the Instituto de Investigaciones en Ingeniería Eléctrica "Alfredo Desages." He teaches undergraduate introductory courses in control theory and robust control theory. He also teaches a graduate course in system modeling and identification. His research interest include nonlinear system modeling, identification and control, piecewise linear approximation of nonlinear systems, and robust control of nonlinear systems.